Set theory - Winter semester 2016-17

Problems	Prof. Peter Koepke
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Problem 23 (2 points)**.** Prove that $\text{card}(n) = n$ for all $n \in \omega$.

Problem 24 (4 points)**.** Prove the following statements.

(a) Every subset of a finite set is finite.
(b) If a, b are finite, then $a \cup b$ is finite.
(c) Is a, b are finite, then $a \times b$ is finite.
(d) If a is finite, then $\mathcal{P}(a)$ is finite and has size $2^{card(a)}$.

Problem 25 (2 points)**.** Prove in the theory ZF that the well-ordering principle implies the axiom of choice AC.

Problem 26 (4 points)**.** Show that in the theory ZF, the axiom of choice is equivalent to the *Hausdorff Maximality Principle*: for every partial order $(P, \leq) \in V$, there is an inclusion maximal chain X in $(P, \leq)$. I.e. if $Y \supseteq X$ is a chain in $(P, \leq)$, the $Y = X$.

Problem 27 (8 points)**.** We consider the *linear order topology* τ on the class Ord of ordinals. The basic open sets of τ are the intervals

$$(\alpha, \beta) := \{\gamma \in \text{Ord} \mid \alpha < \gamma < \beta\}$$

for $\alpha < \beta$ in Ord and

$$[0, \alpha) := \{\gamma \mid \gamma < \alpha\}$$

for α in Ord. A set of ordinals is *open* with respect to τ if there is a cardinal μ and a sequence $\langle U_\alpha \mid \alpha < \mu \rangle$ of basic open sets with $x = \bigcup_{\alpha < \mu} U_\alpha$. Prove the following statements.

(a) A set x of ordinals is closed with respect to τ if and only if $\sup(y) = \cup y \in x$ for all subsets $y \subseteq x$.
(b) For all $\alpha \in \text{Ord}$, the set α is (covering) compact with respect to τ if and only if $\alpha = 0$ or α is a successor ordinal.

(c) For any ordinal α, the function $f_\alpha\colon \mathrm{Ord} \to \mathrm{Ord}$, $f_\alpha(\beta) = \alpha + \beta$ is continuous with respect to τ.

(d) The function $f^\beta\colon \mathrm{Ord} \to \mathrm{Ord}$, $f^\beta(\alpha) = \alpha + \beta$ is not continuous with respect to τ for $\beta = 1$ and for $\beta = \omega$.

Due Friday, December 2, before the lecture.